Econ 802

Answers to Midterm 2

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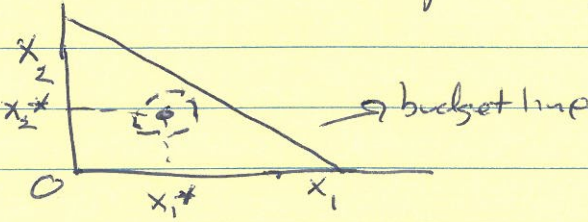
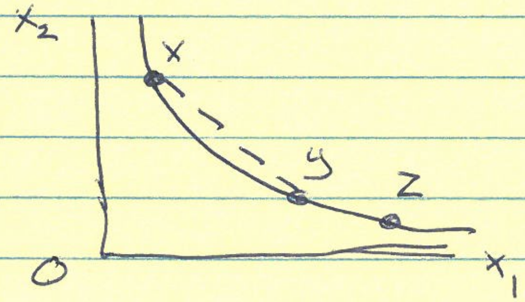
1. (a) A consumer has locally non-satiated preferences if for every bundle x in the consumption set X and every $\epsilon > 0$ there is some bundle $y \in X$ with $|x-y| < \epsilon$ such that $y \succ x$ (strict preference). When this is true, an optimal bundle x^* cannot have $px^* < m$. If it did we could choose $\epsilon > 0$ small enough that all $y \in X$ with $|x-y| < \epsilon$ are affordable. This implies some affordable y is strictly preferred to x^*, which contradicts the optimality of x^*. On a graph:



$\Rightarrow$ some point y in the neighborhood with radius $\epsilon > 0$ is preferred to x^*

(b) A consumer has strictly convex preferences if for any $x, y,$ and $z \in X$ with $x \neq y$, $x \succsim z$, and $y \succsim z$, it is true that $tx + (1-t)y \succ z$ for all $0 < t < 1$. Suppose x and y are both optimal with $x \neq y$. Then x and y are on the same indifference curve. Choose any z on this indifference curve so $x \sim y \sim z$. Because the budget set is convex any consumption bundle $tx + (1-t)y$ is feasible (note that x and y must both be feasible if they are optimal. But then $tx + (1-t)y$ is strictly preferred to x and y if $0 < t < 1$. This contradicts optimality for x and y, and implies uniqueness.

On a graph:
Any point on the dashed line is affordable if x and y are affordable, and such points are strictly preferred to both x and y.



(c) The Marshallian demands maximize utility subject to the budget constraint. Thus they solve

$$\max\ u(x) \text{ subject to } x \geq 0 \text{ and } px = m$$

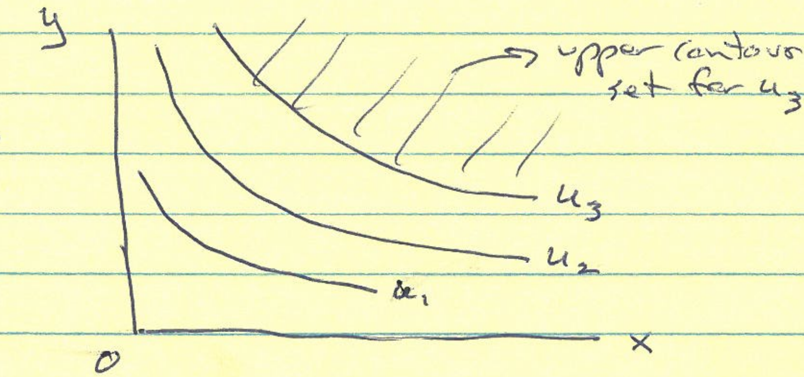
where $x(p,m)$ satisfies $px = m$ with equality due to local non-satiation and is unique by strict convexity. The solution $x(p,m)$ satisfies $x(tp, tm) = x(p,m)$ for all $t > 0$ because this does not change the set of feasible bundles and it does not affect the preference ordering. As a result, whatever bundle $x(p,m)$ solved the problem initially will still solve the problem for (tp, tm).

2. (a) Along an indifference curve we have $y = u - 1 + e^{-ax}$. The slope of the indifference curve is $\left.\frac{dy}{dx}\right|_{u=\text{constant}} = -ae^{-ax} < 0$ and the second derivative is

$$\left.\frac{d^2y}{dx^2}\right|_{u=\text{constant}} = a^2 e^{-ax} > 0.$$

Thus the indifference curves look like this:

Yes, the utility function is strictly quasi-concave because the upper contour sets are strictly convex (see shaded area)



2(b) We want to solve $\max\ 1-e^{-ax}+y$ subject to $px+qy=m$ (Note: we have local non-satiation so the budget constraint holds with equality.)
We don't bother with a Lagrangean because we can substitute $y=\frac{m-px}{q}$ into the utility function:

So $\max\ 1-e^{-ax}-\frac{px}{q}+\frac{m}{q}$
Assuming an interior solution, this gives the FOC
$$ae^{-ax}-\frac{p}{q}=0 \Rightarrow x^*=-\frac{1}{a}\ln\left(\frac{p}{aq}\right)$$
Then go back to the budget constraint to get
$$y^*=\frac{m}{q}+\left(\frac{p}{aq}\right)\ln\left(\frac{p}{aq}\right)$$

Due to the linearity of $u(x,y)$ in y, the Hessian is
$$\frac{\partial^2 u(x^*,y^*)}{\partial x^2}=\begin{bmatrix}-a^2e^{-ax^*} & 0\\ 0 & 0\end{bmatrix}$$
This is negative semi-definite but <u>not</u> negative definite, so the necessary SOC holds but the sufficient SOC does not (note that the restriction $p\cdot h=0$ is irrelevant)

(c) With $(p,q,m)>0$, the origin $x=y=0$ cannot be a solution. So there are 3 possible cases: (i) $x^*=0$, $y^*=\frac{m}{q}>0$; (ii) $x^*>0$ and $y^*>0$; or (iii) $x^*=\frac{m}{p}>0$, $y^*=0$.

Case (i) occurs when $ae^{-ax}-\frac{p}{q}\leq 0$ at $x=0$, or $\boxed{a\leq\frac{p}{q}}$
Case (ii) occurs when $ae^{-ax}=\frac{p}{q}$ for some $x\in(0,\frac{m}{p})$
This implies $ae^{-ax}<\frac{p}{q}$ at $x=\frac{m}{p}$ or $ae^{-\frac{am}{p}}<\frac{p}{q}$
it also implies $ae^{-ax}>\frac{p}{q}$ when $x=0$, or $\frac{p}{q}<a$
so in this case $\boxed{ae^{-\frac{am}{p}}<\frac{p}{q}<a}$
Case (iii) occurs when $ae^{-ax}\geq\frac{p}{q}$ at $x=\frac{m}{p}$ or $\boxed{\frac{p}{q}\leq ae^{-\frac{am}{p}}}$
This covers all possible cases.

3. (a) Roy's identity gives $x_1(p,m) = \frac{\alpha m}{p_1}$ and $x_2(p,m) = \frac{(1-\alpha)m}{p_2}$.
Shephard's lemma gives

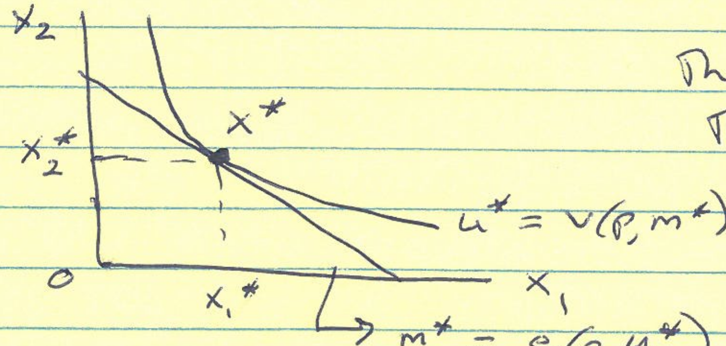
$h_1(p,u) = e^u \alpha p_1^{\alpha-1} p_2^{1-\alpha}/K$ and

$h_2(p,u) = e^u (1-\alpha) p_1^{\alpha} p_2^{-\alpha}/K$.

Substitute $v(p,m) = \ln\left[\frac{mK}{p_1^{\alpha} p_2^{1-\alpha}}\right]$ into the Hicksian demands to get

$h_1[p, v(p,m)] = \frac{\alpha m}{p_1}$ and $h_2[p, v(p,m)] = \frac{(1-\alpha)m}{p_2}$

Thus the Marshallian demands at (p,m) are equal to the Hicksian demands at $[p, v(p,m)]$. On a graph:



The same point x^* that solves the problem of maximizing utility when income is m^* also minimizes expenditure when utility is u^*.

(b) Initially Jim is getting the utility level $v(p,m)$. In order to be equally well off, he needs the new income level m' to satisfy $v(q,m') = v(p,m)$.
From $v(p,m) = \ln\left[\frac{mK}{p_1^{\alpha} p_2^{1-\alpha}}\right]$ this implies

$$\frac{m'}{q_1^{\alpha} q_2^{1-\alpha}} = \frac{m}{p_1^{\alpha} p_2^{1-\alpha}} \Rightarrow \boxed{m' = \frac{m q_1^{\alpha} q_2^{1-\alpha}}{p_1^{\alpha} p_2^{1-\alpha}}}$$

In the special case where $q = tp$ for $t > 0$, this gives $m' = tm$. This is just the zero-degree homogeneity of the indirect utility function: if we multiply all prices <u>and</u> income by the same scalar $t > 0$ the optimal bundle does not change and therefore the utility level does not change.

3. (c) We know that aggregate Marshallian demands can be written as a function of aggregate income M only if all indirect utility functions are in the Gorman form $a_i(p) + b(p) m_i$ for $i = 1 \ldots n$. Otherwise, the distribution of income will affect market demand. The indirect utility function $z(p, m_i) = \dfrac{K m_i}{p_1^{\alpha} p_2^{1-\alpha}}$ is in the Gorman form with $a_i(p)$ and $b(p) = \dfrac{K}{p_1^{\alpha} p_2^{1-\alpha}}$. However, this only works if $b(p)$ is identical for all $i = 1 \ldots n$, which requires $\alpha_i = \alpha$ to be identical for all i. If α_i is different across consumers then we have to write $b_i(p) = \dfrac{K}{p_1^{\alpha_i} p_2^{1-\alpha_i}}$ and aggregation will not work.

4. (a) The budget constraint is $px \le wH$ or $px \le w(T-L)$. This yields $px + wL \le wT$. To get the demand for leisure we maximize $u(x, L)$ subject to $px + wL = m$ (assume non-satiation). This gives the Marshallian demand $L(p, w, m)$. Then substitute $m = wT$ to get $L(p, w, wT)$. To find the effect of w (the price of leisure) on L, write

$$\frac{dL}{dw} = \frac{\partial L(p,w,wT)}{\partial w} + \frac{\partial L(p,w,wT)}{\partial m}\frac{dm}{dw} \quad \text{where } \frac{dm}{dw} = T.$$

Slutsky says $$\frac{\partial L(p,w,wT)}{\partial w} = \frac{\partial h[p,w,v(p,w,wT)]}{\partial w} - \frac{\partial L(p,w,wT)}{\partial m} L(p,w,wT)$$

Substituting this into the previous line gives

$$\frac{dL}{dw} = \underbrace{\frac{\partial h[p,w,v(p,w,wT)]}{\partial w}}_{\substack{\text{substitution effect} \\ \le 0}} + \underbrace{\frac{\partial L(p,w,wT)}{\partial m}}_{\substack{\text{normal good} \\ \Rightarrow\ > 0}} \underbrace{\left[T - L(p,w,wT)\right]}_{\substack{\ge 0 \text{ from} \\ \text{time constraint.}}}$$

Thus we can have $\frac{dL}{dw} > 0$ even though the substitution effect is negative, as long as leisure is a normal good and she is a large enough net supplier of time to the labor market.

4. (b) Assume $u(x, L)$ is differentiable and strictly quasi-concave, so the utility max problem has a unique solution. If (x^*, L^*) is a solution, it must satisfy the FOC for utility max:

$$\frac{\partial u(x^*, L^*)}{\partial x} = \lambda p$$

$$\frac{\partial u(x^*, L^*)}{\partial L} = \lambda w$$

and $$px^* + wL^* = wT$$

we want to find "inverse demands" $p(x, L)$ and $w(x, L)$ using these conditions.

Multiply the first equation by x^* and the second by L^* to get

$$\frac{\partial u(x^*, L^*)}{\partial x} x^* = \lambda p x^*$$

$$\frac{\partial u(x^*, L^*)}{\partial L} L^* = \lambda w L^*$$

Then sum these to get $\frac{\partial u(x^*, L^*)}{\partial x} x^* + \frac{\partial u(x^*, L^*)}{\partial L} L^* = \lambda \{px^* + wL^*\}$
$= \lambda w T$

Now we can solve for $\lambda = \frac{1}{wT} \left\{ \frac{\partial u}{\partial x} x^* + \frac{\partial u}{\partial L} L^* \right\}$

Substituting for λ in the FOC and solving for (p, w) gives

$$p = \frac{wT \frac{\partial u(x^*, L^*)}{\partial x}}{\frac{\partial u(x^*, L^*)}{\partial x} x^* + \frac{\partial u(x^*, L^*)}{\partial L} L^*} \qquad w = \frac{wT \frac{\partial u(x^*, L^*)}{\partial L}}{\frac{\partial u(x^*, L^*)}{\partial x} x^* + \frac{\partial u(x^*, L^*)}{\partial L} L^*}$$

The problem here is that we can solve for the ratio $\frac{p}{w}$, but w cancels in the second equation so we can't solve for w. Furthermore, the second equation only holds if

$$\frac{\partial u(x^*, L^*)}{\partial x} x^* + \frac{\partial u(x^*, L^*)}{\partial L} L^* = T \frac{\partial u(x^*, L^*)}{\partial L}$$

If (x^*, L^*) happens to satisfy this equation then we can find a solution for (p, w) that will work. However, (x^*, L^*) may not satisfy this equation, and if not, then we cannot find any (p, w) such that the FOC holds for (x^*, L^*). The problem here is that w has two roles: it is both the price of leisure and it determines the income $m = wT$, so w and m are not independent parameters as they would be in a standard utility max problem.

4. (c) There are two general approaches: Hicksian and functional separability (you only need to discuss one of these, but here I will describe both).

Hicksian: restrict prices of food goods so that $p = tp_0$ where p_0 is constant and $t > 0$ is a scalar. Thus relative prices of food items never change. The original problem is to max $u(x, L)$ subject to $px + wL = wT$. Rewrite the constraint as $tp_0x + wL = wT$ or $tX + wL = wT$ where $X \equiv p_0x$. Now define the indirect utility function $v(t, w, m) \equiv \max u(x, L)$ subject to $tp_0x + wL = m$. Finally, define a new direct utility function using (using t as the price of the composite good X).

$$g(X, L) \equiv \min v(t, w, 1)$$

subject to $tX + wL = 1$ where the minimization is with respect to (t, w). Now it is possible to solve the problem of maxing $g(X, L)$ subject to $tX + wL = wT$ as in part (a).

Functional: restrict preferences so that $u(x,L)$ can be written in the form $u(x,L) = g[h(x), L]$ where $\frac{\partial g}{\partial h} > 0$. As long as we are making assumptions, let's also assume that $h(x)$ is linearly homogeneous. Then we can define the indirect utility function

$$X(p, m^x) \equiv \max h(x) \text{ subject to } px = m^x$$

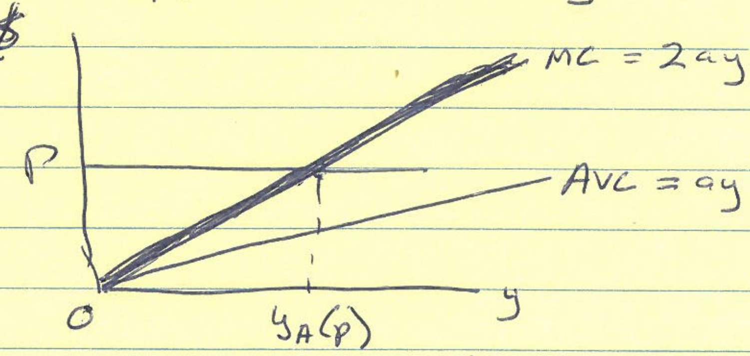
where m^x is total expenditure on all food items. Due to the linear homogeneity of $h(x)$ we know that this function has the form $X(p, m^x) = v(p) m^x$. Maximizing $h(x)$ is a necessary condition for maximizing $u(x,L)$ so we can write the overall direct utility as

$g(X, L)$ with the budget constraint $m^x + wL = wT$

or $\frac{1}{v(p)} X + wL = wT$

Here $\frac{1}{v(p)}$ functions as the price of the composite food good X. The rest of the analysis is the same as part (a).

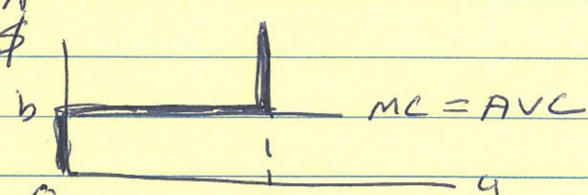
5. (a) For type A: $MC = 2ay$ and $AVC = ay$ so on a graph



(heavy line = supply curve)

It is always true that $P = MC > AVC$ so the firm never shuts down. For all $p > 0$ we have $p = 2ay$ and thus $y_A(p) = \frac{P}{2a}$

For type B: $MC = AVC = b$ so on a graph



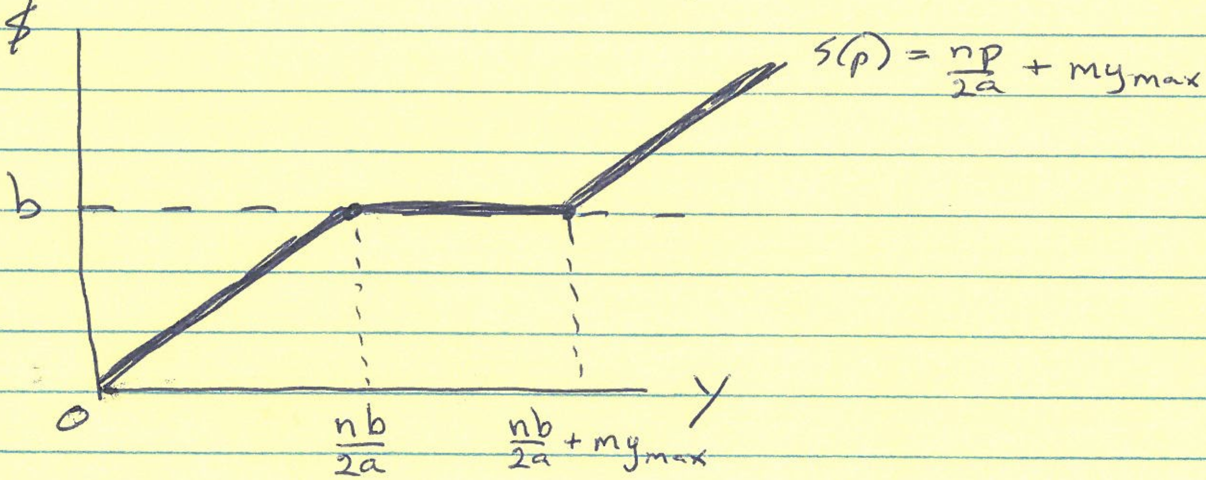
Thus $p < b \Rightarrow y_B(p) = 0$

$p = b \Rightarrow y_B(p) \in [0, y_{max}]$

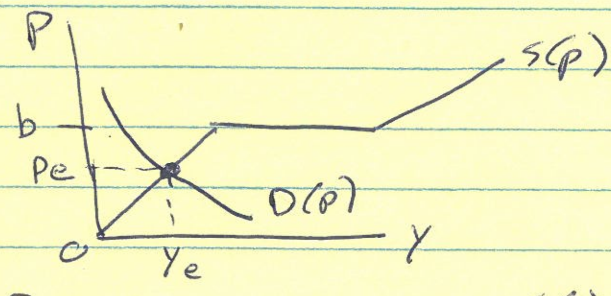
$p > b \Rightarrow y_B(p) = y_{max}$.

(b) For firms of type A as a group, supply is always
$$S_A(p) = n y_A(p) = \frac{np}{2a}$$

For firms of type B as a group, supply is
$S_B(p) = 0$ if $p < b$;
$S_B(p)$ can be anything between 0 and $m y_{max}$
(The individual firms are indifferent and we can't say anything specific about how much each produces) if $p = b$;
and $S_B(p) = m y_{max}$ if $p > b$.
Using $S(p) = S_A(p) + S_B(p)$, we obtain

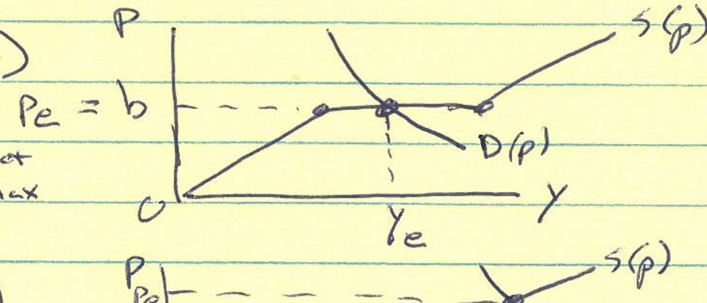


(c) Case (i)



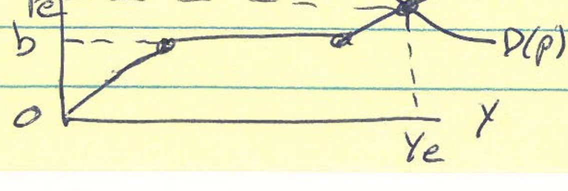
Only type A has $y > 0$ because the equilibrium price has $p_e < b \Rightarrow y_B(p_e) = 0$.

Case (ii)
(note that B firms cannot all have ymax here)



type B must have $S_B(p_e) > 0$ because the output of the A firms is not enough to reach Ye.

Case (iii)



type B firms produce ymax and would like to have $y_B > y_{max}$ because $P_e > MC = AVC$.